MATH 2230BC Tutorial Note 8

1. Find the Taylor series for
$$f(z) = \frac{e^z}{1-z}$$
around $z = 0$. Give the radius of convergence.

   Solution We start by writing the Taylor series for each of the factors and then multiply them out.
$$\begin{aligned} f(z) &= (1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots)(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + (1+1)z + (1+1+\frac{1}{2})z^2 + (1+1+\frac{1}{2}+\frac{1}{3!})z^3 + \cdots \end{aligned}$$
   The biggest disk around $z = 0$ where f is analytic is $|z| < 1$. Therefore, by Taylor's theorem, the radius of convergence is $R = 1$.

2. Find the Taylor series for
$$f(z) = \frac{1}{1-z}$$
around $z = 5$. Give the radius of convergence.

   Solution We have to manipulate this into standard geometric series form
$$\begin{aligned} f(z) &= \frac{1}{-4(1+\frac{z-5}{4})} = -\frac{1}{4}\frac{1}{1-(-\frac{z-5}{4})} \\ &= -\frac{1}{4}\Big(1 - \frac{z-5}{4} + (\frac{z-5}{4})^2 - (\frac{z-5}{4})^3 + \cdots\Big) \end{aligned}$$
   So the series converges when $\frac{|z-5|}{4} < 1$, i.e. $|z-5| < 4$.

3. Expand the rational function
$$f(z) = \frac{1+2z^2}{z^3+z^5}$$
around $z = 0$.

   Solution Note that f has a singularity at 0, so we can't expect a convergent Taylor series expansion. We rewrite $f(z)$ as
$$f(z) = \frac{1}{z^3}\frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}(2 - \frac{1}{1+z^2}).$$
   Since
$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = 1 - z^2 + z^4 - z^6 + \cdots$$

Then

$$f(z) = \frac{1}{z^3}(2 - 1 + z^2 - z^4 + \cdots) = \frac{1}{z^3} + \frac{1}{z} + \sum_{=0}^{\infty}(-1)^n z^{2n+1}.$$

4. Suppose that $C = |z| = \frac{1}{2}$ oriented counterclockwise. Compute

$$\int_C \frac{1+2z}{z^2+z^3} dz.$$

Solution Note that

$$\begin{aligned} \frac{1+2z}{z^2+z^3} &= \frac{1+2z}{z^2}\frac{1}{1+z} \\ &= (\frac{1}{z^2} + \frac{2}{z})\sum_{n=0}^{\infty}(-z)^n \\ &= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots . \end{aligned}$$

Then

$$\int_C \frac{1+2z}{z^2+z^3} dz = \int_C (\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots)dz = 2\pi i.$$